ELSEVIER
Topology and its Applications 100 (2000) 93–101
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Scott spaces and sober spaces

Elias David

23 Lingwood Rd, London E5 9BN, UK

Received 3 January 1996; received in revised form 16 March 1998

Abstract

The categories of sober spaces and generalized (i.e., not necessarily T_0) Scott spaces are adjoint. We define *upper*- and *lower-sober* spaces; lower-sober spaces are a reflective subcategory of TOP that is intermediate to that of sober spaces. We establish why the sobrifications of Alexandrov spaces and of certain Scott spaces described by Mislove (1981) are Scott. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Adjoint functors; Scott sobrification; Upper-sober spaces; Lower-sober spaces; Lower sobrification

AMS classification: 06B35; 54F05

1. Introduction

The closely related S, S_1, and Scott topologies on a quasi- or pre-ordered set are defined in terms of its directed sets, bounded directed sets and directed sets with sups, respectively; alternatively, a topological space is defined to be S, S_1, or Scott if its topology coincides with the respective S, S_1, or Scott topology based on the specialization pre-order.

We proceed to define *sub-Scott*/etc. spaces; these include sober and T_1 spaces and are convenient stepping-stones towards obtaining adjoint functors from each of the categories of Scott/etc. spaces to the categories of coframes and of sober spaces. The fix-points of these functors are the sober Scott spaces.

Abstracting the T_0 condition from the definition of sobriety, we define a space to be *sauber* if every join-prime of the cotopology (the collection of closed sets) is the closure of a singleton. This definition in turn falls into two halves: we define a space to be *upper-sauber* if every such join prime is the closure of a directed set in the specialization order and *lower-sauber* if the closure of every directed set is the closure of a singleton.

A *lower-sober* space is a T_0 lower-sauber space and these spaces form a reflective subcategory of TOP that is intermediate to that of sober spaces. The lower sobrification of an

PII: S0166-8641(98)00089-3

upper-sauber space is its sobrification. We find conditions on an upper-sauber space for its sobrification to be Scott which are met by both Alexandrov and upper-sauber Scott spaces. This explains why the sobrifications of certain Scott spaces described by Mislove are Scott.

2. Scott and related cotopologies on quasi-ordered sets

We define the Scott cotopology (collection of closed sets) on a quasi-ordered set $(Z, \leqslant)$ as follows: for $X \subseteq Z$ write $\sum X$ for the set of least upper bounds of X, i.e., $\sum X = X^u \cap X^{ul}$. Thus if $(Z, \leqslant)$ is a partially ordered set then $\sum X$ is either void or a singleton and $\sum X$ is nonvoid iff $\bigvee X$ exists.

Definition 1. We define a subset W to be *Scott-closed* if for every directed $D \subseteq W$ such that $\sum D \neq \emptyset$ we have $D^{ul} \subseteq W$.

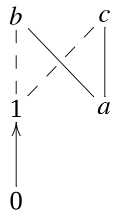
We also define two closely related cotopologies as follows:

(1) A subset W is S_1-*closed* if for every directed $D \subseteq W$ such that $D^u \neq \emptyset$, we have $D^{ul} \subseteq W$.

(2) A subset W is S-*closed* if for every directed $D \subseteq W$ we have $D^{ul} \subseteq W$.

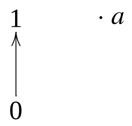
Clearly S-closed sets are S_1-closed and S_1-closed sets are Scott-closed. For up-complete sets, i.e., for which every directed set has a sup, these three coincide.

Example 1. Let $Z = [0, 1) \cup \{a, b, c\}$ with the partial order as shown in the diagram.



For $x \in [0, 1)$, we have $x < b$ and $x < c$. The subset $[0, 1)$ is Scott-closed but not S_1-closed.

Example 2. Let $Z = \{a\} \cup [0, 1)$ as shown.



The subset $[0, 1)$ is S_1-closed but not S-closed.

It is known that a function $f : M \to Z$ between *posets* is Scott-continuous if it satisfies any of the following equivalent conditions:

(1) The inverse image of each Scott-closed set of Z is Scott-closed.

(2) The inverse image of each principal ideal of Z is Scott-closed.

(3) f preserves joins of directed subsets of M.

Similar conditions hold for functions between quasi-ordered sets and all three cotopologies.

Theorem 2. *Let $f:M \to Z$ be a function between quasi-ordered sets. The following conditions on f are equivalent*:

(1) *The inverse image of each Scott/S/S_1-closed subset of Z is a similar subset of M.*
(2) *The inverse image of each principal ideal of Z is respectively Scott/etc. closed.*
(3) *For each directed subset D of M, as appropriately such that $\sum D \neq \emptyset$, or $D^u \neq \emptyset$, we have $f(D^{ul}) \subseteq (fD)^{ul}$.*

In the "Scott" case we have a fourth equivalent condition:

(4) *For each directed subset D of M such that $\sum D \neq \emptyset$ we have*

$$f\Bigl(\sum D\Bigr) \subseteq \sum (fD).$$

Proof. We prove only the "Scott" case of this and all subsequent theorems.

(1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Let $D \subseteq M$ be directed, $\sum D \neq \emptyset$ and $a \in D^{ul}$ and $x \in (fD)^u$. We show that $fa \leqslant x$.

Now $fD \subseteq \{x\}^k$ so $D \subseteq f^{-1}\{x\}^k$ which is Scott-closed. Hence $D^{ul} \subseteq f^{-1}\{x\}^k$ but $a \in D^{ul}$ so $fa \in \{x\}^k$.

(3) $\Rightarrow$ (4) We note that both (3) and (4) imply that f is order preserving so $f(D^u) \subseteq (fD)^u$. Hence

$$f\Bigl(\sum D\Bigr) = f(D^u \cap D^{ul}) \subseteq (fD)^u \cap (fD)^{ul} = \sum (fD).$$

(4) $\Rightarrow$ (1) Let $W \subseteq Z$ be Scott-closed and $D \subseteq f^{-1}W$ and $\sum D \neq \emptyset$. We prove that D^{ul}, which is $(\sum D)^l$, is a subset of W. Now $fD \subseteq W$ and fD is directed, also $f(\sum D) \subseteq \sum (fD)$ so $\sum (fD) \neq \emptyset$. Hence $(fD)^{ul} \subseteq W$ but $(fD)^{ul} = (\sum fD)^l$ and also $\sum (fD) \subseteq (\sum fD)^l$. Hence

$$f\Bigl(\sum D\Bigr) \subseteq \sum (fD) \subseteq \Bigl(\sum fD\Bigr)^l = (fD)^{ul} \subseteq W.$$

Thus $\sum D \subseteq f^{-1}W$ which is a lower set so $(\sum D)^l = D^{ul} \subseteq f^{-1}W$. $\square$

The earliest predecessor of this theorem is by Jurgen Schmidt in [8].

We accordingly define Scott, S_1, and S-continuity. S-continuity implies S_1, and S_1 implies Scott. If the domain of the function is up-complete then these three continuities are equivalent.

We thus have three topological spaces for each quasi-ordered set and three corresponding functors from categories of these sets to that of topological spaces. If $\mathcal{S}$ is the Scott/etc. cotopology on $(Z, \leqslant)$, we call $(Z, \mathcal{S})$ the Scott/etc. space of $(Z, \leqslant)$.

3. Scott and sub-Scott spaces

There is another approach to these spaces using the *specialization* (*quasi-*)*order* on topological spaces. Recall that this is defined as $a \leqslant b$ if $a \in \{b\}^k$, the closure of $\{b\}$.

Definition 3. Let $(Z, \mathcal{W})$ be a topological space where Z is the underlying set and $\mathcal{W}$ the cotopology. (We shall invariably specify the cotopology rather than the topology.) We say that $(Z, \mathcal{W})$ is a Scott/S/S_1 space if $\mathcal{W}$ is the same as the respectively Scott/etc. cotopology based on the specialization (quasi-)order.

If the specialization order is *up-complete*, i.e., if for each directed subset D we have $\sum D \neq \emptyset$, then a space is Scott iff it is S_1 and also iff it is S. We call such spaces *up-complete Scott*.

Definition 4. We say that the space $(Z, \mathcal{W})$ is *sub-Scott/S_1/S* if $\mathcal{W}$ is included in the respectively Scott/etc. cotopology based on the specialization (quasi-)order.

Thus every sub-S space is sub-S_1 and every sub-S_1 space is sub-Scott; once again all these coincide if the specialization order is up-complete. If further the space is T_0, i.e., if the specialization order is anti-symmetric, then this becomes the definition of a *monotone convergence space*, as defined in the Compendium [5], a generalization of a sober space.

Lemma 5. *The space $(Z, \mathcal{W})$ is sub-Scott/S_1/S iff for each subset D that is directed in the specialization order and as appropriately where $\sum D$ or $D^u \neq \emptyset$, we have $D^{ul} = D^k$, the closure of D.*

Proof. For any space $(Z, \mathcal{W})$ and $X \subseteq Z$, we have $X^k \subseteq X^{ul}$. This is because $X^{ul} = \bigcap_{a \in X^u} \{a\}^k$ which is closed.

Next, suppose, for example, $(Z, \mathcal{W})$ is sub-Scott. Then for each directed D such that $\sum D \neq \emptyset$ we have D^k to be Scott-closed, hence $D^{ul} \subseteq D^k$.

On the other hand, suppose that for each such D we have $D^{ul} \subseteq D^k$. Then for each $W \in \mathcal{W}$ we show W to be Scott-closed. Let D be directed and $\sum D \neq \emptyset$ and $D \subseteq W$. Then $D^k \subseteq W$, i.e., $D^{ul} \subseteq W$ so W is Scott-closed. □

A well-known property of monotone convergence spaces is in fact true for sub-Scott spaces.

Proposition 6. *Let $(Z_1, \mathcal{W}_1)$ be a sub-Scott/S/S_1 space and $f : (Z_1, \mathcal{W}_1) \to (Z_2, \mathcal{W}_2)$ be a continuous function. Then f is respectively Scott/S/S_1-continuous in the specialization orders.*

Proof. This follows from the second equivalent formulation in Theorem 2. □

Theorem 7. *The category of Scott/etc. spaces is a coreflective subcategory of the category of respectively sub-Scott/etc. spaces.*

Proof. Suppose, for example, $(Z, \mathcal{W})$ is sub-Scott. Let $\mathcal{S}$ be the Scott cotopology based on the specialization order. Then $(Z, \mathcal{S})$ is a Scott space and the identity map $id : (Z, \mathcal{S}) \to (Z, \mathcal{W})$ is continuous. Next, let $(A, \mathcal{E})$ be a Scott space and $f : (A, \mathcal{E}) \to (Z, \mathcal{W})$ be continuous. Then, by the previous proposition, $f : (A, \mathcal{E} \to (Z, \mathcal{S}))$ is continuous. $\square$

4. Sauber spaces

The concept of *sauber* spaces, originally called *Mammy* spaces in [1], is cognate to that of sober spaces.

Definition 8. A space $(Z, \mathcal{W})$ is called *sauber* if for any space $(A, \mathcal{E})$ and coframe map $\theta : \mathcal{W} \to \mathcal{E}$ there exists a (continuous) function $f : A \to Z$ such that θ is the same as f^{-1} factored through the cotopologies. Taking β as the contravariant functor from topological spaces to coframes, this just means that $\theta = \beta f$.

It is shown in [1] that a space $(Z, \mathcal{W})$ is sauber iff for each join-prime P of the coframe $\mathcal{W}$ there exists an $x \in Z$ such that $P = \{x\}^k$. Thus a space is sober iff it is sauber and T_0.

It is also known, cf. the Compendium [5], that for *any* space $(Z, \mathcal{W})$, if D is directed in the specialization order, then D^k is a join-prime of $\mathcal{W}$. We are led to split the definition of sauber spaces into two halves:

Definition 9a. A space $(Z, \mathcal{W})$ is called *upper-sauber* if each join-prime of $\mathcal{W}$ is the closure of a directed set of the specialization order.

Definition 9b. A space $(Z, \mathcal{W})$ is called *lower-sauber* if for each such directed subset D, there exists an $x \in Z$ such that $\{x\}^k = D^k$.

Clearly a space is lower-sauber iff it is up-complete and sub-Scott. Adding the T_0 condition we define a *lower-sober space*: this is precisely what the Compendium calls a *monotone convergence space*.

5. The lower sobrification of a space

Let $(Z, \mathcal{W})$ be a space with associated specialization order $\leqslant$. Let $\mathcal{Y} \subseteq \mathcal{W}$ be the smallest subset of $\mathcal{W}$—a complete lattice!—that is closed under directed joins and contains the closures of all singletons of Z, i.e., all the principal ideals. Then $\mathcal{Y}$ is up-complete, contains the closures of all directed subsets, and in turn is a subset of $\mathcal{P}$, the collection of join-primes of $\mathcal{W}$. As $\mathcal{P}$ too is closed under directed joins of $\mathcal{W}$, the inclusion function from $\mathcal{Y}$ to $\mathcal{P}$ is Scott-continuous. $\mathcal{P}$ is the underlying set of the space $sob(Z, \mathcal{W})$.

We topologise $\mathcal{Y}$ as a subspace of $sob(Z, \mathcal{W})$ and denote the space by $\mathcal{Y}^*$. The specialization orders on $\mathcal{Y}$ and $\mathcal{P}$ coincide with inclusion. We see that the sobrification map $f:(Z,\mathcal{W})\to sob(Z,\mathcal{W})$ factors through the space $\mathcal{Y}^*$ and so the inclusion map from $\mathcal{Y}^*$ into $sob(Z,\mathcal{W})$ is a sobrification of $\mathcal{Y}^*$ (see diagram).

$$\begin{tikzcd}
(Z,\mathcal{W}) \arrow[rr, "f"] \arrow[dr, "g"'] & & sob(Z,\mathcal{W}) \\
& \mathcal{Y}^* \arrow[ur, "i"'] &
\end{tikzcd}$$

Theorem 10. *The space $\mathcal{Y}^*$ is lower-sober and the above restriction $g:(Z,\mathcal{W})\to\mathcal{Y}^*$ has the property that for any lower-sober space $(M,\mathcal{N})$ and continuous function $h:(Z,\mathcal{W})\to(M,\mathcal{N})$ there exists a unique continuous function $\theta:\mathcal{Y}^*\to(M,\mathcal{N})$ such that $\theta\circ g=h$.*

$$\begin{tikzcd}
(Z,\mathcal{W}) \arrow[r, "h"] \arrow[d, "g"'] & (M,\mathcal{N}) \arrow[rr, "f"] & & sob(M,\mathcal{N}) \\
\mathcal{Y}^* \arrow[ur, dashed, "\theta"'] \arrow[rr, "i"'] & & sob(Z,\mathcal{W}) \arrow[ur, dashed, "\alpha"'] &
\end{tikzcd}$$

Proof. $\mathcal{Y}^*$ is certainly up-complete and T_0. We prove it sub-Scott.

Let $D\subseteq\mathcal{Y}$ be directed. Then $i\,D$ is directed in $\mathcal{P}$ and that space is sub-Scott so $(i\,D)^k=(i\,D)^{ul}$. Since i is a subspace map $i^{-1}[(i\,D)^k]=D^k$ and since i is Scott-continuous $D^{ul}\subseteq i^{-1}[(i\,D)^{ul}]$.

We now prove the universal mapping property. Let $f:(M,\mathcal{N})\to sob(M,\mathcal{N})$ be the sobrification map. Certainly there exists a unique continuous α in the diagram above such that $\alpha\circ i\circ g=f\circ h$. Since both α and f preserve directed joins and f is an order imbedding, there exists a *function* θ such that $f\circ\theta=\alpha\circ i$ and further since f is a subspace map θ is continuous. Hence $f\circ\theta\circ g=\alpha\circ i\circ g$ and $\theta\circ g=h$, again because f is a subspace map. Further, $\alpha=sob\,\theta$. Finally if $\zeta\circ g=h$ for some ζ then $f\circ\zeta=sob\,\zeta\circ i$ and $f\circ\zeta\circ g=sob\,\zeta\circ i\circ g=f\circ h=\alpha\circ i\circ g$ so $\alpha=sob\,\zeta=sob\,\theta$ and $\zeta=\theta$. □

Result 11. The category of lower-sober spaces is a reflective subcategory of TOP and the lower-sobrification of the space $(Z,\mathcal{W})$ is the function $g:(Z,\mathcal{W})\to\mathcal{Y}^*$ described above.

We notice that if $(Z,\mathcal{W})$ is upper-sauber then $\mathcal{Y}=\mathcal{P}$ and so $\mathcal{Y}^*$ is $sob(Z,\mathcal{W})$.

We should mention that using order-theoretic considerations that are too long to go into here and shall be described in another paper, we prove that the lower-sobrifications of many spaces including Alexandrov, S, S_1 and Scott spaces are Scott. Related results on the sobrification of upper-sauber spaces are given later in this paper using topological arguments.

6. Sober spaces and Scott spaces

The categories of sub-Scott/etc. spaces form convenient nodes for adjoint functors in various directions. We have already met the coreflectors to the corresponding subcategories of respectively Scott/etc. spaces. On the other hand they contain the category of sober spaces so have reflectors to this subcategory. Again, for the same reason, they are convenient "codomains" for the functor *cospec* from the category of coframes. Since the composition of adjoint functors is adjoint, we have adjoint functors between the categories of Scott/etc. spaces and each of the categories of sober spaces and of coframes (the latter pair are contravariant).

Let $\mathcal{A}$ be the category of Scott/etc. spaces and $\mathcal{M}$ be that of sober spaces. Let $\alpha : \mathcal{A} \to \mathcal{M}$ be the "Scottification" functor and $\beta : \mathcal{M} \to \mathcal{A}$ be the adjoint sobrification functor. Clearly a Scott/etc. space $(Z, \mathcal{W})$ is a fix-point of the natural transformation $\lambda : id_A \to \beta\alpha$ iff it is sober and conversely a sober space $(A, \mathcal{E})$ is a fix-point of the natural transformation $\mu : \alpha\beta \to id_M$ iff it is Scott.

In general $\beta\alpha(Z, \mathcal{W})$ is not sober, nor is $\alpha\beta(A, \mathcal{E})$ generally Scott. Given a Scott/etc. space $(Z, \mathcal{W})$, consider the sequence of spaces $(Z, \mathcal{W}), \alpha(Z, \mathcal{W}), \beta\alpha(Z, \mathcal{W}), \ldots,$ and similarly consider the sequences of spaces beginning with a sober space $(A, \mathcal{E})$. Does such a sequence arrive at a fix-point? And in how many steps? The answer is that every sequence that *does* arrive at a fix-point reaches it in at most one step. For example, let us begin with a Scott space $(Z, \mathcal{W})$. Since $\alpha(Z, \mathcal{W})$ is always a retract of $\alpha\beta\alpha(Z, \mathcal{W})$, if one is both sober and Scott then so is the other.

7. Scott sobrifications of upper-sauber spaces

It is neither necessary nor sufficient that a space should be Scott for its sobrification to be Scott. Johnstone [5] gives an example of a lower-sober Scott space whose sobrification is not Scott. Alexandrov spaces—those spaces whose closed sets are precisely the lower sets in the specialization order—are not generally Scott but their sobrifications are invariably Scott. They are, however, upper-sauber. Mislove [7] gives an example of a class of Scott spaces whose sobrifications are Scott and has communicated a wider class with this property. These spaces are upper-sauber too. We shall find necessary and sufficient conditions for the sobrification of an upper-sauber space to be Scott that implies that the sobrifications of both Alexandrov spaces and upper-sauber Scott spaces are Scott.

Lemma 13. *Let $(Z, \mathcal{W})$ be an upper-sauber space, let $\mathcal{P} \subseteq \mathcal{W}$ be the collection of join-primes as previously and put the hull-kernel cotopology on $\mathcal{P}$. Let $\mathcal{N} \subseteq \mathcal{P}$ be Scott-closed in $(\mathcal{P}, \subseteq)$. Then $\mathcal{N}$ is closed iff $\bigcup \mathcal{N}$ is a closed subset of Z.*

Proof. Consider the sobrification function $\lambda : Z \to \mathcal{P}$ defined $\lambda(x) = \{x\}^k$. Since

$$\lambda^{-1}(\mathcal{N}) = \bigcup \mathcal{N},$$

if $\mathcal{N}$ is closed so is $\bigcup \mathcal{N}$. Again, $(\bigcup \mathcal{N})^k = \bigcup (\mathcal{N})^k$ because

$$\mathcal{N}^k = \left\{ P \in \mathcal{P} \mid P \subseteq \left(\bigcup \mathcal{N} \right)^k \right\}.$$

Now suppose $\mathcal{N}$ is *not* closed. Then there exists some $P \in \mathcal{N}^k - \mathcal{N}$. Since the space is upper-sauber $P = D^k$ for some directed $D \subseteq Z$. Now

$$D^k = \bigvee_{x \in D} \lambda(x).$$

Since $\mathcal{N}$ is Scott-closed there exists $x \in D$ such that $\lambda(x)$, i.e., $\{x\}^k \notin \mathcal{N}$. Now if $x \in \bigcup \mathcal{N}$ there exists a P such that $\{x\}^k \subseteq P \in \mathcal{N}$ hence $\{x\}^k \in \mathcal{N}$ again as $\mathcal{N}$ is Scott-closed. □

Result 14. The sobrification of every Alexandrov space is Scott.

Proof. The union of any subset of $\mathcal{P}$ in Lemma 13 above is a lower set of Z. □

Proposition 15. *The sobrification of every upper-sauber S/S_1-/Scott space is Scott.*

Proof. Let $(Z, \mathcal{W})$ be Scott and $\mathcal{N} \subseteq \mathcal{P}$ be as in Lemma 13. Then $\lambda^{-1}(\mathcal{N})$ is Scott-closed and so closed. □

We now examine the wider class of Scott spaces communicated by Mislove. Let $(Z, \leqslant)$ be a poset for which $\Downarrow x$ is directed and $\bigvee \Downarrow x = x$ for all x. We show that its Scott space is upper-sauber and consequently that its sobrification is Scott.

Let P be a join-prime of the Scott cotopology. Clearly $P = (\bigcup_{x \in P} \Downarrow x)^k$ and we now prove that $\bigcup_{x \in P} \Downarrow x$ is directed. Let A be a finite subset of this set. Then for each $a \in A$ we have $\Uparrow a \cap P \neq \emptyset$ so $P \nsubseteq \Uparrow a^c$ which is closed. Hence $P \nsubseteq \bigcup_{a \in A} (\Uparrow a)^c$ and there exists some $x \in P \cap \bigcap_{a \in A} \Uparrow a$. Hence $A \subseteq \Downarrow x$ which is directed so for some $y \in \Downarrow x$ we have $y \in A^u$. But $\Downarrow x \subseteq \bigcup_{x \in P} \Downarrow x$ and so this set is directed.

We present some open questions:

(1) If a space and its sobrification are both Scott, is the space upper-sauber?

The Compendium [5] shows that every core-compact T_0 Scott space that is a complete lattice (in its specialization order) is sober.

(2) Is the same true for every core-compact T_0 Scott space that is *up-complete*?

(3) Is it true for every *locally compact* T_0 Scott space that is up-complete?

(4) Is it true for every up-complete T_0 Scott space with a *compact open base* for the topology?

Notes.

(1) For previous work on S-cotopologies see, for example, Erné [3] where it is called the *Scott* cotopology.

(2) The concept of S_1 spaces was put forward by Reinhold Heckemann during discussions comparing Scott and S spaces; he also pointed out that, as for Scott spaces, products coincide with products of partially ordered sets.

(3) These three cotopologies are particular cases of certain *ideals* as defined in [2]. Let $(Z, \leqslant)$ be a quasi-ordered set and $\mathcal{A}$ a collection of its subsets. A subset W is called an $\mathcal{A}$-*ideal* if (1) W is a lower set, i.e., if $a \leqslant b \in W$ then $a \in W$, and (2) for each

$A \in \mathcal{A}$ s.t. $A \subseteq W$ we have $A^{ul} \subseteq W$. The collection of $\mathcal{A}$-ideals forms a closure system for Z. This definition differs slightly from that used by Doctor [3] and Schmidt [8]. Let $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$ be, respectively the directed sets, directed sets with upper bounds, and directed sets with sups. Then the S, S_1 and Scott cotopologies are, respectively the $\mathcal{D}$, $\mathcal{D}_1$ and $\mathcal{D}_2$-ideals.

Acknowledgements

This paper could never have been (re)typed without the instruction and assistance of Bernard Kestelman, son of the late Prof. H. Kestelman.

Dr. M.B. Smyth is thanked for his assistance in redrafting this paper in line with the referee's comments. Our discussions also led to the discovery of lower-sobrifications.

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